# TRANSPORT OF A HIGH-CURRENT RELATIVISTIC ELECTRON BEAM BY MAGNETIC MULTIPOLES

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*With the aid of moments of the distribution function a system of differential equations is obtained to describe the dynamics of a Gaussian high-current electron beam in magnetic fields with quadrupole and octupole symmetries. Results of its numerical solution are reported.*

**Introduction.** The method of moments makes it possible to describe, in analytical form, the dynamics of a high-current relativistic electron beam (HCREB) in external electromagnetic fields regardless of the form of the particle distribution function provided that the acting forces are linear [1, 2]. In this case, an infinite chain of the momental equations can be closed by using the conservation law of root-mean-square beam emittance as a consequence of the Liouville theorem being already valid for the second-order moments. Nonlinearities of the external field and the space charge field cause an increase in emittance. Use of the method of averages in the case of a small statistical nonlinearity allows a one-dimensional systems to be analyzed [3]. Below we consider nonlinearities of the 3rd degree owing to which the motions in the $x$- and $y$-directions turn to be coupled. Allowance for the quadrupole and octupole symmetries of external magnetic fields and properties of the two-dimensional Gaussian distribution allows the chain of equations to be closed with accuracy to squares of the nonlinear terms by adding nine linear differential equations for the fourth moments to the equations for the second moments.

**Momental Equations.** We will consider a relativistic beam of electrons with charge $e$ and mass $m$ moving along the $z$-axis with velocity $v_0$ in magnetic fields with quadrupole and octupole components. For instance, transportation of charged beams by multiplets of quadrupole and octupole lenses: the former are used for beam focusing; the latter for correction of spherical aberration. Another example is transport in quadrupole channels with allowance for the edge effects of lenses due to the nonlinear octupole component of a field. A beam is considered to be a high-current one but its current $I$ does not exceed $I_A = \beta\gamma c^3/e$, where $\gamma = 1/\sqrt{1-\beta^2}$, $\beta^2 c^2 = (\dot{x}^2 + \dot{y}^2 + \dot{z}^2)$. From the condition $I \ll I_A$ it follows that the length of betatron oscillations of electrons is considerably larger than the beam dimension; therefore, particle motions in the longitudinal and transverse directions are decoupled [4]. Since for $x' = \dot{x}/\dot{z}$ and $y' = \dot{y}/\dot{z}$ (the dot designates the derivative with respect to $t$) $|x'| \ll 1$ and $|y'| \ll 1$ are valid, then in a linear approximation for small terms $x'$ and $y'$ we have $v \approx \dot{z} \approx v_0$ for the velocity of an arbitrary particle of the beam. It is assumed that in cross-section the beam form is nearly elliptical, i.e., distortions of the elliptical lines of a constant level of charge density due to nonlinearities of the intrinsic and external fields are effects of second-order smallness.

Using the initial and central moments of the distribution function [1], we write equations for the time variation of the coordinates of the mass center of the beam $\bar{x}$ and $\bar{y}$ and its lateral dimension $\tilde{x} = (\overline{x^2})^{1/2}$ and $\tilde{y} = (\overline{y^2})^{1/2}$ as follows:

$$\gamma m \ddot{\bar{x}} = \overline{F_x^{\text{field}}} + \overline{F_x^{\text{beam}}}, \quad \gamma m \ddot{\bar{y}} = \overline{F_y^{\text{field}}} + \overline{F_y^{\text{beam}}}; \tag{1}$$

$$\ddot{\tilde{x}}\tilde{x} + \dot{\tilde{x}}\dot{\tilde{x}} = \frac{1}{\gamma^2 m^2}\overline{p_x^2} + \frac{1}{\gamma m}\overline{xF_x^{\text{field}}} + \frac{1}{\gamma m}\overline{xF_x^{\text{beam}}},$$

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$$\ddot{\tilde{y}}\tilde{y} + \dot{\tilde{y}}\dot{\tilde{y}} = \frac{1}{\gamma^2 m^2}\overline{p_y^2} + \frac{1}{\gamma m}\overline{yF_y^{\rm field}} + \frac{1}{\gamma m}\overline{yF_y^{\rm beam}}\,; \tag{2}$$

$$\gamma^2 m^2 \dot{\rho}_x = 2\gamma m\,\overline{\dot{x}\,F_x^{\rm field}} + 2\gamma m\,\overline{\dot{x}F_x^{\rm beam}}\,,\quad \gamma^2 m^2 \dot{\rho}_y = 2\gamma m\,\overline{\dot{y}\,F_y^{\rm field}} + 2\gamma m\,\overline{\dot{y}F_y^{\rm beam}}\,, \tag{3}$$

where $p_x = \gamma m\dot{x}$, $p_y = \gamma m\dot{y}$ and $\rho_x = (1/\gamma^2 m^2)\overline{p_x^2}$, $\rho_y = (1/\gamma^2 m^2)\overline{p_y^2}$.

Note that expressions (3) can be also obtained from the equations for the envelopes if for the root-mean-square emittances $\varepsilon_x^2 = \overline{x^2}\,\overline{p_x^2} - \overline{xp_x}^2$ and $\varepsilon_y^2 = \overline{y^2}\,\overline{p_y^2} - \overline{yp_y}^2$ the laws of their variation are written in the form:

$$\dot{\varepsilon}_x = \frac{1}{\varepsilon_x}(\tilde{x}^2\,\overline{\dot{x}F_x} - \tilde{x}\dot{\tilde{x}}\,\overline{xF_x})\,,\quad \dot{\varepsilon}_y = \frac{1}{\varepsilon_y}(\tilde{y}^2\,\overline{\dot{y}F_x} - \tilde{y}\dot{\tilde{y}}\,\overline{yF_y})\,.$$

Using the expressions for the vector potential components $A^{\rm field}(x, y, z)$ of the magnetic quadrupole and octupole and considering that $\varphi^{\rm field} = 0$, we can write the components of the focusing magnetic field in the form:

$$B_x = g^{\rm qu}(z)\,y + g^{\rm oc}(z)\,(y^3 - 3x^2y)\,,\quad B_y = g^{\rm qu}(z)\,x + g^{\rm oc}(z)\,(3y^2x - x^3)\,,$$

$$B_z = 0\,,$$

where $g^{\rm qu}(z)$, $g^{\rm oc}(z)$ are the quadrupole and octupole gradients, respectively.

Then the components of the force acting on an arbitrary particle of the beam from the side of the external field are:

$$\begin{aligned} F_x^{\rm field} &= -\,ev_0\,[g^{\rm qu}(z)\,x + g^{\rm oc}(z)\,(3y^2x - x^3)]\,,\\ F_y^{\rm field} &= +\,ev_0\,[g^{\rm qu}(z)\,y - g^{\rm oc}(z)\,(3x^2y - y^3)]\,. \end{aligned} \tag{4}$$

A procedure for calculating the potentials of the field of a space charge is discussed in the next paragraph; the transverse components of the force are as follows:

$$\begin{aligned} F_x^{\rm beam} &= \frac{e}{\gamma^2}(C_1(z)\,x + C_2(z)\,x^3 + C_3(z)\,y^2x)\,,\\ F_y^{\rm beam} &= \frac{e}{\gamma^2}(C_4(z)\,x + C_5(z)\,x^3 + C_6(z)\,x^2y)\,. \end{aligned} \tag{5}$$

With regard to expressions (4) and (5) the equations for $\tilde{x}$, $\tilde{y}$, $\rho_x$, $\rho_y$ can be written as

$$\ddot{\tilde{x}}\tilde{x} + \dot{\tilde{x}}\dot{\tilde{x}} = \rho_x - \frac{ev_0}{\gamma m}(g^{\rm qu}\tilde{x}^2 - g^{\rm oc}\tilde{x}^4 + 3g^{\rm oc}\,\overline{x^2y^2}) + \frac{e}{\gamma^3 m}(C_1\tilde{x}^2 + C_2\tilde{x}^4 + C_3\,\overline{x^2y^2})\,, \tag{6}$$

$$\ddot{\tilde{y}}\tilde{y} + \dot{\tilde{y}}\dot{\tilde{y}} = \rho_y + \frac{ev_0}{\gamma m}(g^{\rm qu}\tilde{y}^2 + g^{\rm oc}\tilde{y}^4 - 3g^{\rm oc}\,\overline{x^2y^2}) + \frac{e}{\gamma^3 m}(C_4\tilde{y}^2 + C_5\tilde{y}^4 + C_6\,\overline{x^2y^2})\,, \tag{7}$$

$$\dot{\rho}_x = -\,2\frac{ev_0}{\gamma m}(g^{\rm qu}\tilde{x}\dot{\tilde{x}} - g^{\rm oc}\tilde{x}^3\dot{\tilde{x}} + 3g^{\rm oc}\,\overline{xy^2\dot{x}}) + 2\frac{e}{\gamma^3 m}(C_1\tilde{x}\dot{\tilde{x}} + C_2\tilde{x}^3\dot{\tilde{x}} + C_3\,\overline{xy^2\dot{x}})\,, \tag{8}$$

$$\dot{\rho}_y = +2\frac{ev_0}{\gamma m}(g^{qu}\,\overline{\tilde{y}\dot{\tilde{y}}} + g^{oc}\,\overline{\tilde{y}^3\dot{\tilde{y}}} - 3g^{oc}\,\overline{yx^2\dot{y}}) + 2\frac{e}{\gamma^3 m}(C_4\,\overline{\tilde{y}\dot{\tilde{y}}} + C_5\,\overline{\tilde{y}^3\dot{\tilde{y}}} + C_6\,\overline{yx^2\dot{y}})\,. \tag{9}$$

With regard to expressions (4) and (5) the expressions for $\ddot{\bar{x}}$, $\ddot{\bar{y}}$ acquire the form

$$\ddot{\bar{x}} = -\frac{ev_0}{\gamma m}(g^{qu}\,\bar{x} - g^{oc}\,\overline{x^3} + 3g^{oc}\,\overline{xy^2}) + \frac{e}{\gamma^3 m}(C_1\,\bar{x} + C_2\,\overline{x^3} + C_3\,\overline{xy^2})\,, \tag{10}$$

$$\ddot{\bar{y}} = -\frac{ev_0}{\gamma m}(g^{qu}\,\bar{y} - g^{oc}\,\overline{y^3} - 3g^{oc}\,\overline{x^2y}) + \frac{e}{\gamma^3 m}(C_4\,\bar{y} + C_5\,\overline{y^3} + C_6\,\overline{x^2y})\,. \tag{11}$$

We now introduce a new independent variable $\tau$: $d\tau = (v_0/S)dt$, $d\tau = dz/S$, $\tau \in [\tau_0;\ \tau_0 + P/S]$, where $S$ is a period of the focusing structure and $P$ is the transport path length, and rewrite Eqs. (6)-(11) (a dot now denotes the derivative with respect to $\tau$; $\rho_x = \overline{\dot{x}^2}$, $\rho_y = \overline{\dot{y}^2}$):

$$\overline{\ddot{\tilde{x}}\tilde{x}} + \overline{\dot{\tilde{x}}\dot{\tilde{x}}} = \rho_x - \frac{eS^2}{\gamma m\beta c}(g^{qu}\,\overline{\tilde{x}^2} - g^{oc}\,\overline{\tilde{x}^4} + 3g^{oc}\,\overline{x^2y^2}) + \frac{eS^2}{\gamma^3 m\beta^2c^2}(C_1\,\overline{\tilde{x}^2} + C_2\,\overline{\tilde{x}^4} + C_3\,\overline{x^2y^2})\,, \tag{6a}$$

$$\overline{\ddot{\tilde{y}}\tilde{y}} + \overline{\dot{\tilde{y}}\dot{\tilde{y}}} = \rho_y + \frac{eS^2}{\gamma m\beta c}(g^{qu}\,\overline{\tilde{y}^2} + g^{oc}\,\overline{\tilde{y}^4} - 3g^{oc}\,\overline{x^2y^2}) + \frac{eS^2}{\gamma^3 m\beta^2c^2}(C_4\overline{\tilde{y}^2} + C_5\overline{\tilde{y}^4} + C_6\overline{x^2y^2})\,, \tag{7a}$$

$$\dot{\rho}_x = -2\frac{eS^2}{\gamma m\beta c}(g^{qu}\,\overline{\tilde{x}\dot{\tilde{x}}} - g^{oc}\,\overline{\tilde{x}^3\dot{\tilde{x}}} + 3g^{oc}\overline{xy^2\dot{x}}) + 2\frac{eS^2}{\gamma^3 m\beta^2c^2}(C_1\overline{\tilde{x}\dot{\tilde{x}}} + C_2\overline{\tilde{x}^3\dot{\tilde{x}}} + C_3\overline{xy^2\dot{x}})\,, \tag{8a}$$

$$\dot{\rho}_y = +2\frac{eS^2}{\gamma m\beta c}(g^{qu}\,\overline{\tilde{y}\dot{\tilde{y}}} - g^{oc}\,\overline{\tilde{y}^3\dot{\tilde{y}}} - 3g^{oc}\overline{yx^2\dot{y}}) + 2\frac{eS^2}{\gamma^3 m\beta^2c^2}(C_4\overline{\tilde{y}\dot{\tilde{y}}} + C_5\overline{\tilde{y}^3\dot{\tilde{y}}} + C_6\overline{yx^2\dot{y}})\,, \tag{9a}$$

$$\ddot{\bar{x}} = -\frac{eS^2}{\gamma m\beta c}(g^{qu}\,\bar{x} - g^{oc}\,\overline{x^3} + 3g^{oc}\,\overline{xy^2}) + \frac{eS^2}{\gamma^3 m\beta^2c^2}(C_1\bar{x} + C_2\overline{x^3} + C_3\overline{xy^2})\,, \tag{10a}$$

$$\ddot{\bar{y}} = +\frac{eS^2}{\gamma m\beta c}(g^{qu}\,\bar{y} + g^{oc}\,\overline{y^3} - 3g^{oc}\,\overline{x^2y}) + \frac{eS^2}{\gamma^3 m\beta^2c^2}(C_4\bar{y} + C_5\overline{y^3} + C_6\overline{x^2y})\,. \tag{11a}$$

This system of equations is incomplete, since it constitutes undetermined moments of the third $\overline{xy^2}$, $\overline{yx^2}$, $\overline{x^3}$, $\overline{y^3}$ and fourth order $\overline{x^2y^2}$, $\overline{x^2y\dot{y}}$, $\overline{y^2x\dot{x}}$.

For actual transport channels for high-current beams the quadratic terms of the transverse coordinates in Eqs. (6a)-(9a) are considerably larger than the terms of the fourth order. Then the system of the equations for the fourth moments turns to be closed with an accuracy to terms of the second-order infinitesimal. Using the following designations:

$$b_1 = \overline{x^2y\dot{y}},\quad b_2 = \overline{y^2x\dot{x}},\quad b_3 = \overline{x^2\dot{y}^2},\ b_4 = \overline{\dot{x}^2y^2},\ b_5 = \overline{\dot{x}^2y\dot{y}},\quad b_6 = \overline{\dot{y}^2x\dot{x}}\,,$$

$$b_7 = \overline{x^2y^2}\,,\quad b_8 = \overline{x\dot{x}y\dot{y}}\,,\quad b_9 = \overline{\dot{x}^2\dot{y}^2}\,,$$

it can be written in the form:

$$\dot b_1 = b_3 + a_y b_7 + 2b_8\,,\qquad \dot b_4 = 2a_x b_2 + 2b_5\,,\qquad \dot b_7 = 2b_1 + 2b_2\,,$$
$$\dot b_2 = b_4 + a_x b_7 + 2b_8\,,\qquad \dot b_5 = a_y b_4 + 2a_x b_8 + b_9\,,\qquad \dot b_8 = a_x b_1 + a_y b_2 + b_5 + b_6\,, \tag{12}$$
$$\dot b_3 = 2a_y b_1 + 2b_6\,,\qquad \dot b_6 = a_x b_3 + 2a_y b_8 + b_9\,,\qquad \dot b_9 = 2a_x b_5 + 2a_y b_6\,,$$

where

$$a_x(z) = -\frac{eS^2}{\gamma m\beta c} g^{\text{qu}}(z) + \frac{eS^2}{\gamma^3 m\beta^2 c^2} C_1(z)\,;\;\; a_y(z) = \frac{eS^2}{\gamma m\beta c} g^{\text{qu}}(z) + \frac{eS^2}{\gamma^3 m\beta^2 c^2} C_4(z)\,.$$

To determine the unknown third moments in Eqs. (6a)-(11a), we assume that along the entire transport path the charge density in an elliptical cross-section of the beam has a Gaussian distribution with time-dependent parameters (the model of quasi-equilibrium plasma [5], self-similar beams [6], self-similar solutions of the kinetic equation [7]). Then the distribution function of the transverse coordinates in a focusing channel is of the form

$$g(x, y) = \frac{1}{2\pi\sigma_x\sigma_y\sqrt{1-\mu^2}} \exp\left(-\frac{1}{2(1-\mu^2)}\left[\frac{(x-\bar x)^2}{\sigma_x^2} + \frac{(y-\bar y)^2}{\sigma_x^2} - \frac{2\mu}{\sigma_x\sigma_y}(x-\bar x)(y-\bar y)\right]\right). \tag{13}$$

Here, the mean values of $\bar x$, $\bar y$, dispersion $\sigma_x$, $\sigma_y$, and the correlation coefficient $\mu$, which reflects the degree of the linear dependence between $x$ and $y$, are functions of $z$.

For a two-dimensional normal distribution the central moment of any odd order can be expressed in terms of the corresponding moments of the second order, i.e., in terms of the dispersion of random values and the coefficient of correlation between them:

$$\overline{(x-\bar x)^2(y-\bar y)} = 0\,,\quad \overline{(x-\bar x)^3} = 0\,,\quad \overline{(x-\bar x)(y-\bar y)^2} = 0\,,\quad \overline{(y-\bar y)^3} = 0\,,\quad \overline{(x-\bar x)^2(y-\bar y)^2} = \sigma_x^2\sigma_y^2 + 2\mu^2\sigma_x^2\sigma_y^2\,.$$

With this taken into consideration, we express the initial moments of the 3rd order in terms of the central moments of the 3rd order and determine $\overline{xy}$:

$$\overline{x^3} = 3\bar x\overline{x^2} - 2\bar x^3\,,\quad \overline{y^3} = 3\bar y\overline{y^2} - 2\bar y^3\,,$$
$$\overline{x^2y} = -2\bar x^2\bar y + 2\bar x\,\overline{xy} + \overline{x^2}\,\bar y\,,\quad \overline{y^2x} = -2\bar y^2\bar x + 2\bar y\,\overline{xy} + \overline{y^2}\,\bar x\,, \tag{14}$$
$$\overline{xy} = \sqrt{\left(\frac{\overline{x^2y^2} - \overline{x^2}\,\overline{y^2} + 2\,\bar x^2\bar y^2}{2}\right)}\,.$$

The relations for the dispersions and the correlation coefficient are as follows:

$$\sigma_x^2 = \overline{x^2} - \bar x^2\,,\quad \sigma_y^2 = \overline{y^2} - \bar y^2\,,\quad \mu\sigma_x\sigma_y = \overline{xy}\,\bar x\,\bar y\,. \tag{15}$$

Thus, for determination of the root-mean-square dimensions of the beam Eqs. (6a)-(9a) are solved is combination with system (12). Knowing the expressions for $\tilde x$, $\tilde y$ and $\overline{x^2y^2}$, the coordinates of the center of mass of the beam can be found as a solution of Eqs. (10a), (11a) with the use of relations (14).

**Calculation of the Potentials of the Space-Charge Field.** The potentials of the space-charge field on each section element of the transport path $l$ (the $l$ order of the Debye radius, which is determined by the transverse temperature of the beam and particle density being functions of $z$) depend only on the transverse coordinates, the parameters of the distribution function of which do not change along $l$ [5].

To calculate the transverse components of the space charge field of the beam at length $l$, we rewrite (13) in terms of new variables $X = x - \bar{x}$, $Y = (y - \bar{y})\sigma_x/\sigma_y$:

$$g(X, Y) = \frac{1}{2\pi\sigma_x\sigma_x\sqrt{1-\mu^2}} \exp\left(-\frac{1}{2(1-\mu^2)}\left[\frac{X^2 + Y^2 - 2\mu XY}{\sigma_x^2}\right]\right),$$

We now write a solution of the Poisson equation for a Gaussian distribution of charge density in terms of the Green function (for a circle) of the Dirichlet internal boundary-value problem $G(X_0, Y_0, X, Y)$:

$$\varphi^{\text{beam}}(X_0, Y_0) = -\frac{I}{\varepsilon_0 v_0} \int\int dXdY\, g(X, Y) \times$$

$$\times \frac{1}{4\pi} \ln \frac{R^2 + \dfrac{(X^2 + Y^2)(X_0^2 + Y_0^2)}{R^2} - (X^2 + Y^2 + X_0^2 + Y_0^2) + (X - X_0)^2 + (Y - Y_0)^2}{(X - X_0)^2 + (Y - Y_0)^2}, \tag{16}$$

where $\varepsilon_0$ is the vacuum dielectric permittivity; $R = 3\sigma_x$.

The two-dimensional integral (16) cannot be calculated analytically. It must be calculated numerically at each $i$-th node $(X_0^i, Y_0^i)$ of a space grid imposed on the beam cross-section by the Gauss method for hyperrectangles. Integration is over the region $[-3\sigma_x; 3\sigma_x] \times [-3\sigma_x; 3\sigma_x]$. Here, according to the boundary conditions, $\varphi^{\text{beam}} = 0$ beyond this region.

Knowing the potential values at different points of the beam cross-section, we determine its analytical dependence on the transverse coordinates by the least-squares method by approximating, as in [8], using polynomials of the fourth order with respect to $x$ and $y$. Polynomial decomposition of the scalar potential seems to be reasonable, since we can choose, as a geometric model of the beam, an elliptical cylinder (the longitudinal dimension of the beam is far greater than the radial one and the cross-section is assumed to be elliptical).

With regard for elliptical symmetry we have

$$\varphi^{\text{beam}}(x, y, z) = C_{01}(z)\, x^2 + C_{02}(z)\, y^2 + C_{40}(z)\, x^4 + C_{04}(z)\, y^4 + C_{22}(z)\, x^2 y^2. \tag{17}$$

Here, the decomposition factors are constant along $l$ and are recalculated for each step.

In a quasistationary approximation [9], $A_x^{\text{beam}} = A_y^{\text{beam}} = 0$, $A_z^{\text{beam}} = (v_0/c^2)\varphi^{\text{beam}}$; therefore, the force components of the space charge with allowance for the magnetic field of the beam have the form of (5). A comparison of expressions (17) and (5) shows that $C_1 = -C_{20}$, $C_2 = -C_{40}$, $C_4 = -C_{02}$, $C_5 = -C_{04}$, $C_3 = C_6 = -C_{22}$.

**Solution of the System of Momental Equations.** Expression (12) represents a normal linear system of ordinary differential equations. According to the Picard theorem, a fundamental system of solutions exists if the coefficients of (12) are continuous functions. If the gradients of the quadrupole and octupole lenses are prescribed as piecewise continuous functions [9] along the entire transport path and the values of the coefficients $C_i$ ($i = 1, \ldots, 6$) are determined for the given section element $l$, system (12) becomes autonomous and a fundamental system of solutions can be found analytically. With knowledge of $a_x$ and $a_y$ for the given $l$, a solution of system (12) can be written in general form in terms of its eigenvalues and eigenvectors, which are as follows

$$0,\quad \left\{0, 0, -a_y, -a_x, 0, 0, 1, 0, a_x a_y\right\},$$

$$-2\sqrt{a_y},\quad \left\{-\sqrt{a_y}, 0, a_y, -a_x, a_x\sqrt{a_y}, 0, 1, 0, -a_x a_y\right\},$$

$$2\sqrt{a_y},\quad \left\{\sqrt{a_y}, 0, a_y, -a_x, -a_x\sqrt{a_y}, 0, 1, 0, -a_x a_y\right\},$$

$$-2\sqrt{a_x},\quad \left\{0, \frac{1}{a_y\sqrt{a_x}}, \frac{1}{a_x}, -\frac{1}{a_y}, 0, -\frac{1}{\sqrt{a_x}}, -\frac{1}{a_x a_y}, 0, 1\right\},$$

$$2\sqrt{a_x},\quad \left\{0, -\frac{1}{a_y\sqrt{a_x}}, \frac{1}{a_x}, -\frac{1}{a_y}, 0, \frac{1}{\sqrt{a_x}}, -\frac{1}{a_x a_y}, 0, 1\right\},$$

$$-2\sqrt{a_y}-2\sqrt{a_x},\quad \left\{-\frac{a_1(-12a_x-4a_y+a_1^2)}{16a_x(a_x-a_y)}, \frac{a_1(-4a_x-12a_y+a_1^2)}{16a_y(a_x-a_y)}, \frac{(-4a_x-4a_y+a_1^2)}{8a_x},\right.$$

$$\frac{(-4a_x-4a_y+a_1^2)}{8a_y}, -\frac{a_1(-12a_x-4a_y+a_1^2)}{16(a_x-a_y)}, \frac{a_1(-4a_x-12a_y+a_1^2)}{16(a_x-a_y)},$$

$$\left.\frac{(-4a_x-4a_y+a_1^2)}{8a_xa_y}, 1, -\frac{1}{2}a_x-\frac{1}{2}a_y+\frac{1}{8}a_1^2\right\},$$

$$-2\sqrt{a_y}+2\sqrt{a_x},\quad \left\{-\frac{a_2(-12a_x-4a_y+a_2^2)}{16a_x(a_x-a_y)}, \frac{a_2(-4a_x-12a_y+a_2^2)}{16a_y(a_x-a_y)}, \frac{(-4a_x-4a_y+a_2^2)}{8a_x},\right.$$

$$\frac{(-4a_x-4a_y+a_2^2)}{8a_y}, -\frac{a_1(-12a_x-4a_y+a_2^2)}{16(a_x-a_y)}, \frac{a_2(-4a_x-12a_y+a_2^2)}{16(a_x-a_y)},$$

$$\left.\frac{(-4a_x-4a_y+a_2^2)}{8a_xa_y}, 1, -\frac{1}{2}a_x-\frac{1}{2}a_y+\frac{1}{8}a_2^2\right\},$$

$$-2\sqrt{a_y}-2\sqrt{a_x},\quad \left\{-\frac{a_3(-12a_x-4a_y+a_3^2)}{16a_x(a_x-a_y)}, \frac{a_3(-4a_x-12a_y+a_3^2)}{16a_y(a_x-a_y)}, \frac{(-4a_x-4a_y+a_3^2)}{8a_x},\right.$$

$$\frac{(-4a_x-4a_y+a_3^2)}{8a_y}, -\frac{a_3(-12a_x-4a_y+a_3^2)}{16(a_x-a_y)}, \frac{a_3(-4a_x-12a_y+a_3^2)}{16(a_x-a_y)},$$

$$\left.\frac{(-4a_x-4a_y+a_3^2)}{8a_xa_y}, 1, -\frac{1}{2}a_x-\frac{1}{2}a_y+\frac{1}{8}a_3^2\right\},$$

$$2\sqrt{a_y}+2\sqrt{a_x},\quad \left\{-\frac{a_4(-12a_x-4a_y+a_4^2)}{16a_x(a_x-a_y)}, \frac{a_4(-4a_x-12a_y+a_4^2)}{16a_y(a_x-a_y)}, \frac{(-4a_x-4a_y+a_4^2)}{8a_x},\right.$$

$$\frac{(-4a_x-4a_y+a_4^2)}{8a_y}, -\frac{a_4(-12a_x-4a_y+a_4^2)}{16(a_x-a_y)}, \frac{a_4(-4a_x-12a_y+a_4^2)}{16(a_x-a_y)},$$

$$\left.\frac{(-4a_x-4a_y+a_4^2)}{8a_xa_y}, 1, -\frac{1}{2}a_x-\frac{1}{2}a_y+\frac{1}{8}a_4^2\right\},$$

where $a_1 = -2\sqrt{a_y} - 2\sqrt{a_x}$; $a_2 = -2\sqrt{a_y} + 2\sqrt{a_x}$; $a_3 = 2\sqrt{a_y} - 2\sqrt{a_x}$; $a_4 = -2\sqrt{a_y} + 2\sqrt{a_x}$.

Having determined the moments of the 4th order $\overline{x^2y^2}$, $\overline{x^2y\dot{y}}$, $\overline{y^2x\dot{x}}$, we solve Eqs. (6a)-(9a) simultaneously for determination of $\overline{x}$, $\overline{y}$.

Solution of Eqs. (10a)-(11a) in combination with expressions (14) for the 3rd-order moments $\overline{xy^2}$, $\overline{yx^2}$, $\overline{x^3}$, $\overline{y^3}$ allows determination of $\overline{x}$, $\overline{y}$.

Equations (6a)-(11a) are nonlinear, and they can be solved by numerical methods, e.g. by the Runge–Kutta 4th-order method. From the considerations above it is clear that the integration step must not exceed $l$.

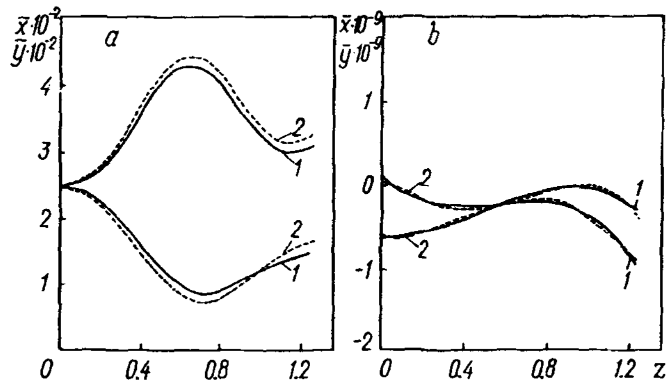


Fig. 1. Root-mean-square dimensions (a) and parameters-centroids (b) of the transported beam: 1) method of moments; 2) TRLIE program. $\tilde{x}$, $\tilde{y}$, $\bar{x}$, $\bar{y}$, z, m.

As an example, we calculate the centroid parameters and root-mean-square dimensions of a beam transported in a focusing-defocusing (FD) channel representing a triplet of nonideal quadrupoles. The length of the transport path is equal to the period of the focusing system and is 1.25 m; the lengths of the triplet lenses are 0.375, 0.5, and 0.375 m, respectively, the quadrupole gradient is $g^{qu} = 0.025$ T/m, and the second derivative (with respect to $z$) of the quadrupole gradient is $g'' = 0.001$ T/m$^3$.

Using the expressions for the components of the vector potential $A^{field}(x, y, z)$ of a magnetic quadrupole with boundary fields [10], we can write the components of the focusing magnetic field with allowance for the nonlinear (octupole) component in the form

$$B_x = g(z)\,y - \frac{1}{12}g''(z)\,y^3 - \frac{1}{4}g''(z)\,x^2 y,$$

$$B_y = g(z)\,x - \frac{1}{12}g''(z)\,x^3 - \frac{1}{4}g''(z)\,y^2 x,$$

$$B_z = 0.$$

Equations (6a)-(11a) retain their form with allowance for $g^{oc}(z) = \pm 1/12 g''$.

Figure 1 shows plots of the variation of the root-mean-square and mean dimensions of the beam obtained by the method described above and the output data of the TRLIE program [11] (numerical simulation of the dynamics of high-current beams using Lie algebra as a tool). For the electron beam we have taken the following initial parameters: current $I = 100$ A, particle energy $E = 1$ MeV, $\tilde{x}(0) = \tilde{y}(0) = 2.5$ cm, $\tilde{x}(0) = \tilde{y}(0) = 0$, and the dispersion of the transverse velocitites is 1% of the longitudinal velocity. As seen, the results of both methods are consistent with each other within an accuracy of 8%.

**Conclusion.** Thus, the presence of quadrupole and octupole symmetry of external magnetic field and the use of the properties of the Gaussian distribution of the transverse coordinates of a high-current relativistic beam of charged particles make it possible to describe analytically its dynamics in a transport channel with allowance for nonlinear effects. The calculation results agree with data obtained by alternative methods for coarse particles.

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## NOTATION

$I_A$, Alvena current; $v_0$, velocity of ordered motion of electrons; $c$, velocity of light in vacuum; $F^{field}$, force of magnetic-focusing element; $F^{beam}$, force of space charge; $g^{qu}$, quadrupole gradient; $g^{oc}$, octupole gradient; FD-channel, transport channel assembled of alternating quadrupole lenses.

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